

26230

NRL Report

Exact Solutions for the Time Constants of an Adaptive Array in Bandlimited Noise

KARL GERLACH

*Airborne Radar Branch
Radar Division*

December 24, 1981



NAVAL RESEARCH LABORATORY
Washington, D.C.

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Report 8542	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) EXACT SOLUTIONS FOR THE TIME CONSTANTS OF AN ADAPTIVE ARRAY IN BANDLIMITED NOISE		5. TYPE OF REPORT & PERIOD COVERED Interim report on a continuing NRL problem.
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Karl Gerlach		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Program Element 62712N 53-0662-0-1 WF12-000-001
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Air System Command Washington, DC 20361		12. REPORT DATE December 24, 1981
		13. NUMBER OF PAGES 24
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Adaptive arrays Radar Antenna		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A technique is presented that yields closed form solutions for the noise analysis of the Applebaum adaptive algorithm. Past researchers have derived results under the assumption that the input noise is stationary and rapidly varying with respect to the output weighting vector of the Applebaum algorithm. In this report, this work has been extended to include noise that is not stationary and can vary at any rate. External noise sources are modelled as continuous state jump Markov processes which results in exact first moment equations for the weighting vector that are solvable. (Continued)		

20. ABSTRACT (Continued)

Specifically, the case where the adaptive algorithm is subjected to a single external noise source is examined in detail. Results in adaptive processing convergence times are presented.

CONTENTS

I.	INTRODUCTION	1
II.	TRANSIENT ANALYSIS OF THE APPLEBAUM ALGORITHM IN WIDEBAND NOISE	2
III.	NOISE MODELS AND STOCHASTIC LINEAR DIFFERENTIAL EQUATIONS	5
	A. Introduction	5
	B. Noise Model for a Continuous State Jump Markov Process	6
	C. Power Spectrum	8
	D. First Moment of a Stochastic Linear Vector Differential Equation	9
IV.	PARTICULAR SOLUTION FOR A SINGLE INTERFERENCE SOURCE	9
	A. Introduction	9
	B. First Moment Formulation	10
	C. Time Constants	13
	D. Exact Solution for First Moment	18
V.	SUMMARY AND CONCLUSIONS	19
VI.	REFERENCES	19

EXACT SOLUTIONS FOR THE TIME CONSTANTS OF AN ADAPTIVE ARRAY IN BANDLIMITED NOISE

I. INTRODUCTION

Adaptive processors are the subject of considerable interest for a variety of applications in the radar and communications fields. The reason for this interest is that adaptive processors can respond automatically to an unknown external noise environment such as radar clutter or man-made interference by steering nulls in the direction of the interfering sources and at the same time maintain a desired signal response.

Widrow [1] has defined an adaptive processor (filter or array) to be a filter that bases its own design (its internal adjustment settings) upon estimated statistical characteristics of the input and output signals. In particular, an adaptive spatial array processor weights the coherent output from each sensor and adds them to form a receiving beam. For an adaptive array these weights may not be constant, but rather can change as a function of the spatial properties of the noise field. Historically, the adaptive filter was first investigated in the early 1960's by Howells [2] and Applebaum [3], the latter of whom discovered the control law which maximizes a generalized signal to noise ratio. A few years later, Widrow [4,5] and his co-workers derived and demonstrated the utility of a least mean square (LMS) algorithm for controlling the weights, and applied their approach to adaptive RF antenna systems. The LMS algorithm was further developed by Griffiths [6] and Frost [7] who found procedures for maintaining a chosen frequency characteristic for an array in a desired direction while nulling out noises coming from other directions. Compton [8,9] and Zahm [10] examined the use of the LMS algorithm as a power equalization technique which allowed the acquisition of weak signals in the presence of strong jamming. Brennan and Reed [11-13] further developed the Applebaum maximum signal to noise ratio (MSN) algorithm by making contributions in the noise analysis of the algorithm and methods of accelerating the convergence of the adapting weights. Gabriel [14] gives an excellent introduction to adaptive arrays.

The subject of this report is the noise analysis of the Applebaum algorithm. A review of the transient analysis involving this algorithm is presented in Section II. In previous analysis [3,11,13,14], the noise field was assumed statistically stationary and fluctuating much more rapidly than the adapting weights. This implied that the instantaneous output weighting vector was independent of the instantaneous input noise. Hence, the expected value of their multiplicative vector product (which occurs in the implementation of the algorithm) could be separated into a product of expected values. This simplified the transient analysis of the algorithm considerably. Bershad [15] extended the noise analysis of the algorithm by lifting the restriction that the output process is independent of the input process. He modelled the input data as white noise processes and the adaptive weight process (output process) as a vector Markov-diffusion process. These assumptions lead to a Fokker-Planck equation for the joint probability density function (p.d.f.) of the adaptive weights from which equations for the first and second moments are derived. However, most often these equations are difficult to solve in closed form and thus measures of the adaptive filter's performance such as the settling time or time constants of the weights, control loop noise, and cancellation ratios of sidelobe noise are not derivable.

In this report, techniques that yield closed formed expressions for the first moment of the weighting vector are presented for when the external noise sources are not necessarily stationary and can vary

at any rate. Hence, solutions for time constants, control loop noise, and sidelobe cancellation ratios can be derived exactly. The technique models the external noise sources as continuous state jump Markov processes [16] that modulate a carrier frequency. Section III discusses the merits of modelling input noise in this manner and shows for example that the spectrum of a colored Gaussian noise process can be replicated by a continuous state jump Markov process. Section III also discusses the theory of linear stochastic differential equations (D.E.) where it is shown that the first moment of the weights is derivable from an integro-differential equation.

Section IV examines the case when the adaptive array is subjected to just one external noise source. The first moment of the adapting weights is derived exactly, and the related time constants are found. In this section, it is also shown that these generalized results reduce to the results that were obtained by past researchers [3,13,14] under the restricting assumption that the input noise varies much more rapidly than the output weighting vector.

II. TRANSIENT ANALYSIS OF THE APPLEBAUM ALGORITHM IN WIDEBAND NOISE

Applebaum [3] laid the foundations of adaptive array processing. Up until his work, pre-1964, the pattern of an antenna array was steered by applying linear phase weighting across the array and shaped by constant amplitudes and phase weighting the output of the array elements. These weights are chosen a priori so as to produce a pattern that is a compromise between resolution, gain, and low sidelobes. Applebaum discovered a relatively simple algorithm which allows the weights to change adaptively in a time-varying noise environment. His "control law" attempts to vary the array weights dynamically such that the steady state signal to noise ratio of the array output in any spatial configuration of noise sources is maximized.

Because by definition, the array is optimized in the steady state, it will take the array weights a certain period of time to approach their steady state optimum values. Thus there are time constants inherent to the adaptive process. In this section, we present the transient analysis of the Applebaum algorithm when the input noise is wideband or equivalently when the input noise is fluctuating much more rapidly than the adjusting weights. For a more detailed presentation, see Gabriel [14].

We will now consider a linearly weighted phased array as seen in Fig. 1. Let $q_1(t), q_2(t), \dots, q_N(t)$ denote the output received by an N -element antenna array, and $Q_T = (q_1, q_2, \dots, q_N)$ where T denotes the transpose. We will assume that Q is the sum of a desired signal vector S and a noise vector V . The noise V could consist of jammers, clutter, atmospheric noise, or internal receiver noise. Let W be a column vector of complex weights that multiplies Q to form the product $W_T Q$. The weights W can be adjusted based on some optimization criteria to enhance the detection of a desired signal embedded in noise. The direction of arrival of the desired signal is assumed known. Let S represent a column vector of relative desired signal phases with

$$S_T = (e^{j\phi_1}, e^{j\phi_2}, \dots, e^{j\phi_N}). \quad (2.1)$$

For a linear array with element spacing d and a signal arriving from an angle ψ with respect to the array normal (see Fig. 2),

$$\phi_n = (2\pi nd/\lambda) \sin \psi$$

where λ is the wavelength and $n = 1, 2, \dots, N$.

The output desired signal power of $W_T Q$ is proportional to $|W_T S|^2$. It is assumed in all foregoing analysis that the input signal S does not contribute significantly to the covariance matrix of the input Q . This assumption is valid for most pulsed radars where the signal duration is short compared to the pulse

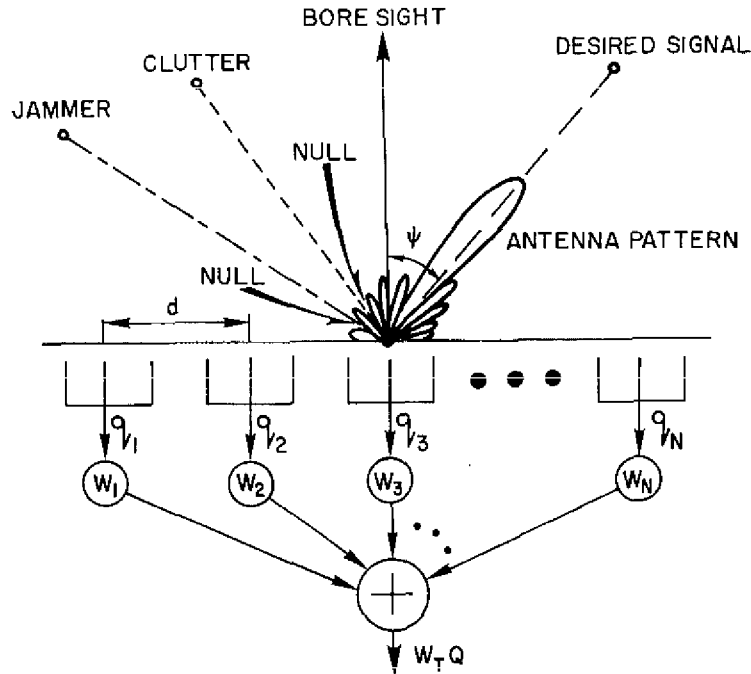


Fig. 1 — Weighted linear array

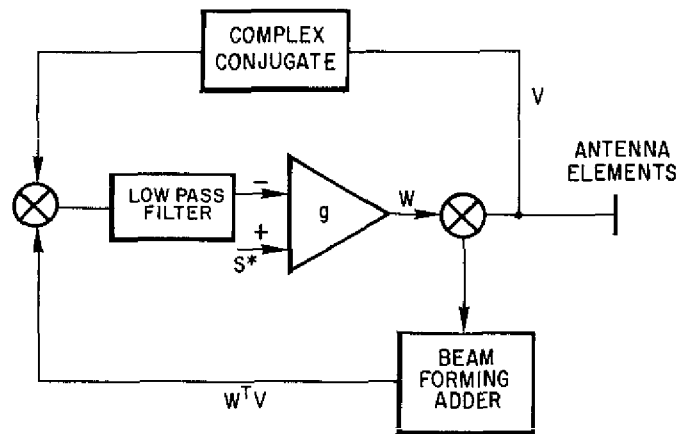


Fig. 2 — Implementation of Applebaum adaptive algorithm

repetition period. It may also be true in cases where the external interference is a man-made jammer and the received power of the jammer is much greater than the received power of the desired signal. For radars, this might occur because of the one-way path of the jammer signal versus the two-way path of the desired signal. The output power P_N of the array then is

$$P_N = E |W_T V|^2 = W_T^* \bar{M} W \quad (2.2)$$

where E denotes expectation and \bar{M} is the covariance of the noise process with elements $\bar{m}_{ij} = E\{v_i^* v_j\}$. The input process is assumed wide sense stationary, and hence \bar{M} is independent of time.

The steady state output signal-to-noise ratio can be written as a proportionality or

$$\frac{S}{N} \sim \frac{|W_T S|^2}{W_T^* \bar{M} W}. \quad (2.3)$$

It is shown in Ref. 3 that the set of weights that maximizes the signal-to-noise ratio is

$$W_{opt} = k \bar{M}^{-1} S^* \quad (2.4)$$

where k is an arbitrary constant.

A functional block diagram of an implementation of the Applebaum adaptive receiving array is seen in Fig. 2. This algorithm attempts to adjust the weighting vector $W(t)$ such that the steady state signal to noise power ratio (S/N) is maximized.

It can be shown [11] that this adaptive array is described by the differential equation

$$\tau \frac{dW}{dt} + (gM + I) W = gS^* \quad (2.5)$$

where τ is the time constant of the low pass filter in each loop, g is amplifier gain in each loop, I is an identity matrix, and

$$M = V^* V_T. \quad (2.6)$$

The matrix M is called the instantaneous covariance matrix with elements that are random variables. Equation (2.5) is a stochastic linear differential equation with the stochastic input M and the stochastic output W .

If the input process is fluctuating much faster than the output process, then we can write

$$E\{M W\} = E\{M\} E\{W\} = \bar{M} \bar{W}. \quad (2.7)$$

This would be the case where the control loop bandwidth is much smaller than the bandwidth of the input noise vector V .

If we take the expected value of both sides of Eq. (2.5) and use Eq. (2.7), then

$$\tau \dot{\bar{W}} + (g\bar{M} + I) \bar{W} = gS^*. \quad (2.8)$$

Because \bar{M} is Hermitian, there exists a unitary transformation P which diagonalizes \bar{M} or

$$P \bar{M} P^{-1} = \Lambda \quad (2.9)$$

where Λ is a $N \times N$ diagonal matrix with elements λ_n equal to the eigenvalues of \bar{M} . If we let $Y = P \bar{W}$ and $Y_T = (y_1, y_2, \dots, y_N)$, then Eq. (2.8) reduces to

$$\tau \dot{Y} + (g\Lambda + I) Y = gR \quad (2.10)$$

where $R = PS^*$, and the initial condition is $Y(0) = P \bar{W}_0$. The solution for y_n is

$$y_n = \frac{r_n}{\lambda_n + \frac{1}{g}} + \left[y_n(0) - \frac{r_n}{\lambda_n + \frac{1}{g}} \right] \exp \left\{ - \frac{g\lambda_n + 1}{\tau} t \right\}, \quad (2.11)$$

where $R_T = (r_1, r_2, \dots, r_N)$. The transient solution of the array weights is obtainable from Eq. (2.11) and the relationship $\bar{W} = P^{-1}Y$. If we examine \bar{W} as $t \rightarrow \infty$, it can be seen from Eq. (2.11) assuming $g\lambda_n \gg 1$ for all n , that $y_n \rightarrow r_n/\lambda_n$ and $\bar{W} \rightarrow \bar{M}^{-1}S^*$ which is the optimal solution that we are seeking.

The time constants associated with Eq. (2.11) are

$$\tau_n = \frac{\tau}{g\lambda_n + 1}; \quad n = 1, 2, \dots, N \quad (2.12)$$

where λ_n are the eigenvalues of \bar{M} . Thus the smallest eigenvalue of \bar{M} , λ_{\min} determines how fast the adaptive array converges to its optimal weights. Each weight w_n is a sum of exponentials whose time constants are τ_n . From Eq. (2.12) it can be seen that to speed up convergence, one can either decrease τ or increase g . However, Brennan et al. [11] showed that this increases the control loop noise power of the adaptive filter.

The brief summary of results presented in this section on the time constants of the Applebaum adaptive algorithm was derived under the restrictions that (1) the input process is varying much more rapidly than the output process and (2) the input process is stationary, i.e., \bar{M} is a constant. In the following sections, the Applebaum algorithm will be analyzed when these restrictions on the input noise process are lifted. We impose some restrictions on our noise model, described in the next chapter. However, this noise model allows us to examine the previously difficult problem of the derivation of time constants for the adaptive weights in a noise environment that is not necessarily wideband.

III. NOISE MODELS AND STOCHASTIC LINEAR DIFFERENTIAL EQUATIONS

A. Introduction

The Applebaum adaptive algorithm expressed by Eq. (2.5) is a stochastic linear differential equation because of the stochastic input vector V . The standard methodology used to study its performance in noise is to derive the first and second moments of the weighting vector W and to analyze performance measures such as time constants and excess noise power due to weight jitter as a function of the parameters of this equation. If we assume that the n th component of V consists of internal receiver noise η_n plus the sum of M external narrow band noise sources, then v_n can be expressed as [14]

$$v_n(t) = \eta_n(t) + \sum_{k=1}^M X_k(t) e^{j(n\phi_k + \xi_k + \Phi_k(t))}; \quad n = 1, \dots, N \quad (3.1)$$

where $\phi_k = (2\pi d/\lambda) \sin \theta_k$, λ is the wavelength of the carrier, d is the antenna element spacing, θ_k is the spatial location angle of the k th source measured from the boresight, $X_k(t)$ is the random amplitude modulation of the k th source, and $\xi_k + \Phi_k(t)$ is the random phase of the k th source which consists of a stochastic component $\Phi_k(t)$ and a nonstochastic component ξ_k . In the analysis to follow, we assume that $\eta_n(t) = 0$, so as to highlight the effect of the external noise sources on the adaptive processor.

The random amplitude modulation of the k th source, $X_k(t)$, is a stochastic process and can be modelled in a variety of ways. Let us just consider a single source with stochastic amplitude modulation equal to $X(t)$. For example, if the amplitude $X(t)$ is modelled as a Markoff diffusion process, then the first moment $\langle W(t) \rangle$ can be formulated as the solution of a nonrandom partial differential equation of the Fokker-Planck type [17]. However, this equation is most often exceedingly difficult to solve because of a diffusion term (2nd derivative with respect to the spatial coordinate). The methodology used in this dissertation is to model the input noise amplitude modulation as a continuous state jump Markoff process (CSJMP) [16]. If $X(t)$ is a CSJMP, then $X(t)$ is discontinuous with the states of $X(t)$ changing abruptly by jumps as illustrated in Fig. 3(a).

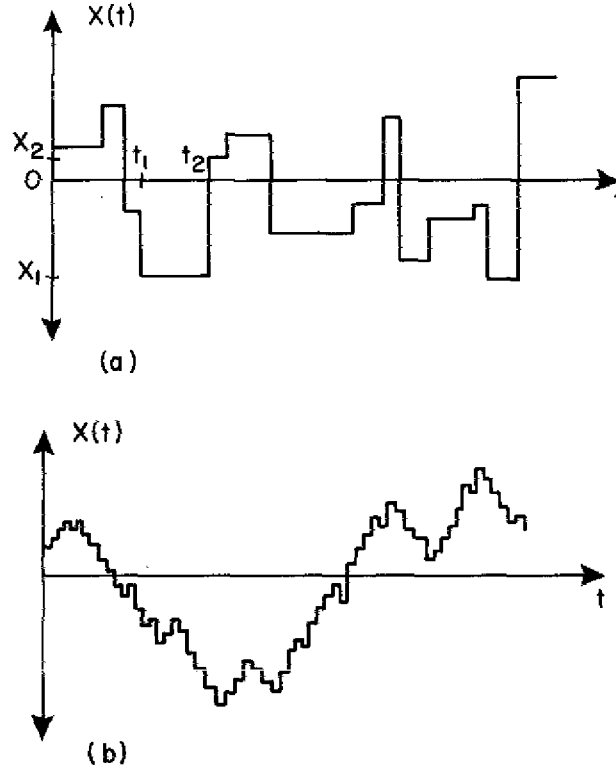


Fig. 3 — Markov jump processes

The amount of time \tilde{T} that $X(t)$ remains in a given state is a random variable and hence can be characterized by a distribution function (for instance, Poisson increments). The successive state of $X(t)$ is determined by a probability distribution. By controlling these two distribution functions, we find that continuous time noise processes can be modelled. For example, if we choose the average value of \tilde{T} small such that $\langle \tilde{T} \rangle^{-1} \gg B$, where B is the bandwidth of the adaptive processor and also choose a probability distribution function which forces a successor state to be highly correlated with its current state, then the sudden jumps of the input process will not affect the output process $W(t)$ and time continuous correlated noise can be simulated as seen in Fig. 3(b).

More importantly though, we find in the following sections that the probability and first moment functions that result when this type of modelling is used can be formulated as the solutions of integro-differential equations which are sometimes readily solvable. Thus an analysis of parametric variations in the adaptive algorithm is possible.

B. Noise Model for Continuous State Jump Markov Process

Kolmogorov [18] and Feller [16] laid the foundations of the theory of jump processes. Both authors were concerned with the purely discontinuous process where the state remains unchanged between jumps. Modern texts which consider the subject are Srinivasan [19], Bharucha-Reid [20], and Prabhu [21]. We introduce the topic by using the terminology of Prabhu.

Let the states (possible values) of the process $X(t)$ belong to the real number line $(-\infty < X < \infty)$. Let the transition distribution function $P(x, t|x_0, t_0)$ be defined by

$$P(x, t|x_0, t_0) = \Pr\{X(t) \leq x | X(t_0) = x_0\}; \quad (t_0 < t) \quad (3.2)$$

and

$$P(x, t | x_0, t) = \mu(x - x_0) \quad (3.3)$$

where $\mu(\cdot)$ is the Heavyside step function.

We require also that $P(x, t | x_0, t_0)$ satisfy the Chapman-Kolmogorov equations and now define a purely discontinuous process as one in which 1) if $X(t) = x$, then a probability $1 - c(x, t)\Delta t + O(\Delta t)$ that there is no change of state ($X(t)$ remains constant during $[t, t + \Delta t)$, and 2) if there is a change of state, then the distribution function of $X(t + \Delta t)$ is given by $\Pi(x, x'; t) + O(\Delta t)$. The function $c(x, t)$ can be interpreted as the "jump rate." We can use $c(x, t)$ to control how often the process switches states: the larger the value of $c(x, t)$, the more rapidly the process is state jumping. The function $\Pi(x, x'; t)$ is actually a conditional probability distribution conditioned on x . The distribution function $\Pi(x, x'; t)$ will be used to control the relative correlation between successive states. Both $c(x, t)$ and $\Pi(x, x'; t)$ determine the correlation function of the process. For our purposes, we assume both functions are temporally independent with $c(x) = c(x, t)$ and $\Pi(x, x'; t) = \Pi(x, x')$ and that both are continuous with $0 \leq c(x) < \infty$. In addition, we assume that $X(t)$ is a second order process (bounded variance).

If we define $p(x, x_0; t, t_0)$ to be the joint p.d.f. of $X(t)$ and $X(t_0)$, and $p(x, t)$ to be the single variable p.d.f. of $X(t)$ at any time t , then it is shown in Ref. 22 that

$$\begin{aligned} \frac{d}{dt} p(x, x_0; t, t_0) = & -c(x)p(x, x_0; t, t_0) \\ & + \int_{-\infty}^{\infty} c(x')\pi(x', x)p(x', x_0; t, t_0)dx' \end{aligned} \quad (3.4)$$

$$\text{I.C.: } p(x, x_0; t_0, t_0) = p(x_0, t_0)\delta(x - x_0)$$

and

$$\frac{d}{dt} p(x, t) = -c(x)p(x, t) + \int_{-\infty}^{\infty} c(x')\pi(x', x)p(x', t)dx' \quad (3.5)$$

$$\text{I.C.: } p(x, t_0) \text{ given}$$

where

$$\pi(x', x) = \frac{\partial}{\partial x} \Pi(x', x). \quad (3.6)$$

The function, $\pi(x', x)$, will be called the transition p.d.f. of the process.

In the following analysis we assume that the process begins in equilibrium. That is, if we assume the existence of the limit and define

$$p_{\infty}(x) = \lim_{t \rightarrow \infty} p(x, t), \quad (3.7)$$

then we will let $p(x, 0)$ or simply, $p_0(x) = p_{\infty}(t)$. The reason for making this assumption is that as $t \rightarrow \infty$, the input process becomes stationary. Hence, we can derive a steady state spectrum. Setting $p_0(x) = p_{\infty}(x)$ simply makes the process stationary at $t = 0$. We use the steady state spectrum of the jump input process to match the spectrum of a time continuous stochastic process in the next section.

Let us formulate $p_\infty(x)$ when $c(x) = c_0$, where c_0 is a constant. Starting with Eq. (3.5), we recognize that as $t \rightarrow \infty$, the time derivative seen in Eq. (3.5) goes to zero. Cancelling c_0 from the resultant equation, we obtain

$$p_\infty(x) = \int_{-\infty}^{\infty} \pi(x', x) p_\infty(x') dx'. \quad (3.8)$$

Now if $\pi(x', x)$ is independent of x' , i.e., the next state of the process, $X(t)$, is independent of the current state, then it is easily seen from Eq. (3.8) that $p_\infty(x) = \pi(x)$ where $\pi(x', x) = \pi(x)$. Thus we see that the steady state p.d.f., $p_\infty(x)$, is simply the transition p.d.f., $\pi(x)$. We call this kind of CSJMP, pseudo white.

Let us consider a CSJMP where the joint p.d.f. of $X(t_0)$ and its next state $X(t)$ is a symmetric function, $p(x', x) = p(x, x')$. The single variable p.d.f., $p(x)$, and $p(x', x)$ are related by the equations: $p(x', x) = \pi(x', x)p(x')$ and

$$p(x) = \int_{-\infty}^{\infty} p(x', x) dx' = \int_{-\infty}^{\infty} \pi(x', x) p(x') dx'. \quad (3.9)$$

Thus comparing Eq. (3.9) with Eq. (3.8), we see that $p_\infty(x) = p(x)$. Therefore to find $p_\infty(x)$, we merely integrate $p(x', x)$ over all of x' . For example, let us define $\pi(x', x)$ to be the Gaussian conditional p.d.f.

$$\pi(x', x) = \frac{1}{\sigma(2\pi(1-\rho^2))^{1/2}} \exp - \frac{1}{2\sigma^2(1-\rho^2)} (x - \rho x')^2. \quad (3.10)$$

If $p(x', x)$ is symmetric, then Eqs. (3.8) and (3.9) imply that

$$p_\infty(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp - \frac{1}{2\sigma^2} x^2. \quad (3.11)$$

Hence $p_\infty(x)$ is a Gaussian p.d.f. We will call noise with a transition p.d.f. given by Eq. (3.10), a colored Gaussian CSJMP. Note that the parameter ρ in Eq. (3.10) can be used to control the degree of correlation between successive states.

If we force the input noise process to begin in equilibrium, i.e., $p_0(x) = p_\infty(x)$, then we can show that the solution, $p(x, t)$ in Eq. (3.5) is simply $p(x, t) = p_\infty(x)$. Hence, the single variable p.d.f. function $p(x, t)$ is independent of time if the noise process begins in equilibrium.

C. Power Spectrum

It is shown in Ref. 22 that the power spectrum of the noise process, $X(t)$, for when the process is in equilibrium, $c(x) = c_0$ where c_0 is a constant, and $\pi(x', x)$ has the form (see Eq. (3.10) for example)

$$\pi(x', x) = \pi(x - \rho x') \quad (3.12)$$

is (also see Fig. 4)

$$S_x(\omega) = \frac{2c_0(1-\rho)\sigma^2}{\omega^2 + c_0^2(1-\rho)^2}. \quad (3.13)$$

Thus we see from Eq. (3.13) that a first order base bandlimited spectrum of a given continuous random process may be matched or modelled by a CSJMP by choosing c_0 and ρ properly. We would seem to have one extra degree of freedom in matching the spectrum. However, other considerations

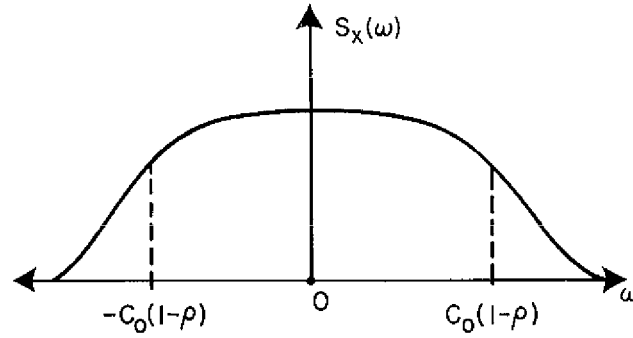


Fig. 4 — Power spectrum

such as for instance making the adaptive receiver insensitive to the sudden jumps have to be weighed in choosing c_0 and ρ . We can show that the average time T that $X(t)$ is in a given state is c_0^{-1} . Thus, it may be desirable to make $c_0 > B$, where B is the bandwidth of the adaptive receiver.

D. First Moment of a Stochastic Linear Vector Differential Equation

Let us consider a stochastic linear vector D.E.

$$\frac{dW}{dt} = F(X(t), t) W(t) + G(X(t), t) \quad (3.14)$$

$$\text{I.C.: } W(0) = W_0.$$

It is shown in Ref. 22 that if $X(t)$ is a CSJMP, then the first moment of the weights $\langle W(t) \rangle$ can be formulated by the following equations:

$$\begin{aligned} \frac{d}{dt} y(x, t) = & \int_{-b}^b c(x) \pi(x', x) y(x', t) dx' - c(x) y(x, t) \\ & + F(x, t) y(x, t) + G(x, t) p(x, t) \end{aligned} \quad (3.15)$$

$$\text{I.C.: } y(x, 0) = p(x, 0) W_0.$$

$$\langle W(t) \rangle = \int_{-\infty}^{\infty} y(x, t) dx. \quad (3.16)$$

Hence, we first find $y(x, t)$ by using Eq. (3.15) and then integrate $y(x, t)$ which yields $\langle W(t) \rangle$. We use Eqs. (3.15) and (3.16) in Section IV to analyze the Applebaum adaptive algorithm when the input noise is a single interfering source generating a CSJMP.

IV. PARTICULAR SOLUTION FOR A SINGLE INTERFERENCE SOURCE

A. Introduction

In this section we consider the case when the adaptive receiver is being interfered with by just one external noise source. We consider this to be the only source of interference, internal or external, so as to highlight the effect of a single noise source on the adaptive filter and to demonstrate the noise analysis techniques presented in the previous section. We represent the noise source's signal as the

scalar $X(t)e^{j\alpha(t)}$, where $X(t)$ is the stochastic amplitude modulation and $\alpha(t)$ is the random phase of the noise source. The amplitude modulation $X(t)$ is modelled as a CSJMP. For a single noise source, the noise power of $X(t)$ is measured at the adaptive receiver and not the noise source's transmitter. Also, the noise source is assumed spatially stationary (it does not move). Furthermore, the noise source's voltage $X(t)$ is assumed narrowband with respect to its carrier frequency so that the input noise vector $V(t)$ is simply $AX(t)e^{j\alpha(t)}$, where $A = (e^{j\phi_1}, e^{j\phi_2}, \dots, e^{j\phi_N})$ indicates the direction of arrival of the input interference and is essentially constant (recall from Eq. (3.1) that ϕ_n is a function of the wavelength of the input signal). Subsection B formulates the solution of the first moment of the weighting vector by using the noise analysis techniques presented in Section III. Subsection C examines in detail the time constants associated with this weighting vector. It is shown that if the input noise bandwidth decreases, then the settling time or time constant of the weighting vector increases. It is also shown that if the input noise bandwidth is finite, then the time constant approaches a positive definite limit as the input noise power becomes large. The exact solution of first moment of the weighting vector is derived in Subsection D.

B. First Moment Formulation

In this section, we formulate the solution of the first moment of the weighting vector of the adaptive algorithm by using the noise analysis techniques described in Section III. We use the assumptions as noted in Subsection A of this chapter. To this end, if we substitute $AX(t)e^{j\alpha(t)}$ for $V(t)$ in the adaptive algorithm (2.5), we obtain the equation

$$\frac{dW}{dt} = - \left[\frac{g}{\tau} X^2(t) A^* A_T + \frac{1}{\tau} I \right] W(t) + \frac{g}{\tau} S^* \quad (4.1)$$

$$\text{I.C. } W(0) = W_0.$$

Note that W is not a function of $\alpha(t)$. Let us just consider the $N \times N$ transition matrix solution [23, p. 131], $\Psi(t)$, of Eq. (4.1) or

$$\frac{d\Psi}{dt} = - \left[\frac{g}{\tau} X^2(t) A^* A_T + \frac{1}{\tau} I \right] \Psi(t) \quad (4.2)$$

$$\text{I.C. } \Psi(0) = I.$$

It can be shown that the solution of Eq. (4.1) is

$$W(t) = \Psi(t) W_0 + \int_0^t \Psi(t - t_1) \frac{g}{\tau} S^* dt_1. \quad (4.3)$$

Taking the expected value of both sides of Eq. (4.3) yields

$$\langle W(t) \rangle = \langle \Psi(t) \rangle W_0 + \int_0^t \langle \Psi(t - t_1) \rangle dt_1 \frac{g}{\tau} S^*. \quad (4.4)$$

Hence if an expression for $\langle \Psi(t) \rangle$ is found, then $\langle W(t) \rangle$ can be evaluated simply by using Eq. (4.4). We use a methodology suggested by Lang and Pickholtz [24], which reduces the number of differential equations to be solved from N^2 (Eq. (4.2)) to just two. The technique takes advantage of the form of the matrix $A^* A_T$. Let us write the solution of Eq. (4.2) in the form

$$\Psi(t) = A^* A_T \psi_1(t) + I \psi_2(t) \quad (4.5)$$

where $\psi_1(t)$ and $\psi_2(t)$ are scalar functions yet to be determined and with initial conditions

$$\psi_1(0) = 0; \psi_2(0) = 1. \quad (4.6)$$

Substituting Eq. (4.5) into Eq. (4.2) yields

$$\frac{d\psi_1}{dt} A^* A_T + \frac{d\psi_2}{dt} I = \left[- \left[\frac{g}{\tau} N X^2(t) + \frac{1}{\tau} \right] \psi_1 + \frac{g}{\tau} X^2(t) \psi_2 \right] A^* A_T - \frac{1}{\tau} \psi_2 I. \quad (4.7)$$

The fact that $A^* A_T A^* A_T = (A_T A^*) (A^* A_T) = N A^* A_T$ was used in deriving Eq. (4.7). Now if we set

$$\frac{d\psi_1}{dt} = - \left[\frac{g}{\tau} N X^2(t) + \frac{1}{\tau} \right] \psi_1 + \frac{g}{\tau} X^2(t) \psi_2 \quad (4.8)$$

$$\frac{d\psi_2}{dt} = - \frac{1}{\tau} \psi_2, \quad (4.9)$$

then we see that the solutions for $\psi_1(t)$ and $\psi_2(t)$ in Eqs. (4.8) and (4.9) will yield the complete solution of the transition matrix $\Psi(t)$ by using Eq. (4.5). Thus, we have reduced the number of D.E.s under consideration from N^2 to just two. Actually the procedure we have used could have been performed more systematically within the context of algebras [25]. The two matrices, $A^* A_T$ and I , are said to be the elements of the algebra, and $\psi_1(t)$ and $\psi_2(t)$ are called structure constants.

Now if we know the first moments of ψ_1 and ψ_2 , then

$$\langle \Psi(t) \rangle = A^* A_T \langle \psi_1(t) \rangle + I \langle \psi_2(t) \rangle. \quad (4.10)$$

It is seen that the structure constant, $\psi_2(t)$, may be solved directly from Eq. (4.9) and the initial condition given in Eq. (4.6) as

$$\psi_2(t) = e^{-t/\tau}. \quad (4.11)$$

Thus we have to consider only the stochastic D.E. given by Eq. (4.8) or

$$\frac{d\psi_1}{dt} = - \left[\frac{g}{\tau} N X^2(t) + \frac{1}{\tau} \right] \psi_1(t) + \frac{g}{\tau} X^2(t) e^{-t/\tau} \quad (4.12)$$

I.C.: $\psi_1(0) = 0$.

Our next step will be to determine $\langle \psi_1(t) \rangle$. If we model the input noise to be a CSJMP with a transition p.d.f., $\pi(x', x)$, and jump rate, c_0 , and set

$$F(X(t), t) = - \frac{g}{\tau} N X^2(t) - \frac{1}{\tau}, \quad (4.13)$$

$$G(X(t), t) = - \frac{g}{\tau} X^2(t) e^{-t/\tau}, \quad (4.14)$$

then we may use Eq. (3.15) to obtain an integro-differential equation for the state dependent mean density function $y(x, t)$. This equation is

$$\frac{d}{dt} y(x, t) = - \left[\frac{gN}{\tau} x^2 + \frac{1}{\tau} + c_0 \right] y(x, t) + c_0 \int_R \pi(x', x) y(x', t) dx' - p(x, t) \frac{g}{\tau} x^2 e^{-t/\tau} \quad (4.15)$$

I.C. $y(x, 0) = 0$,

where $p(x,t)$ is the single variable p.d.f. of $X(t)$ and R is the probability space of x' . The total mean or state independent mean, $\langle \psi_1(t) \rangle$, is found by using Eq. (3.16) or

$$\langle \psi_1(t) \rangle = \int_R y(x,t) dx. \quad (4.16)$$

Let us examine the case when the input noise is pseudo-white and in equilibrium or $\pi(x',x) = \pi(x)$, and $p_0(x) = p_\infty(x) = \pi(x)$.

Thus Eq. (4.15) becomes

$$\begin{aligned} \frac{d}{dt} y(x,t) = & - \left(g \frac{N}{\tau} x^2 + \frac{1}{\tau} + c_0 \right) y(x,t) + c_0 \pi(x) \int_R y(x',t) dx' \\ & - \pi(x) \frac{g}{\tau} x^2 e^{-t/\tau} \end{aligned} \quad (4.17)$$

I.C. $y(x,0) = 0$.

We will now solve Eq. (4.17). Let us define $Y(x,s)$ to be the Laplace transform of $y(x,t)$ and $G(s)$ as

$$G(s) = \int_R Y(x',s) dx'. \quad (4.18)$$

Taking the Laplace transform of both sides of Eq. (4.17) and solving for $Y(x,s)$ yields

$$Y(x,s) = K(x,s) \pi(x) \left(c_0 G(s) - \frac{\frac{g}{\tau} x^2}{s + \frac{1}{\tau}} \right) \quad (4.19)$$

where

$$K(x,s) = \frac{1}{\frac{g}{\tau} N x^2 + \frac{1}{\tau} + c_0 + s}. \quad (4.20)$$

Substituting the expression for $Y(x,s)$ as given in Eq. (4.19) into Eq. (4.18) and solving this linear equation directly for $G(s)$ results in

$$G(s) = - \frac{g}{\tau} \cdot \frac{1}{s + \frac{1}{\tau}} \cdot \frac{\int_R x^2 K(x,s) \pi(x) dx}{1 - c_0 \int_R K(x,s) \pi(x) dx}. \quad (4.21)$$

The expression for $G(s)$ in Eq. (4.21) may now be substituted into Eq. (4.19) to find $Y(x,s)$.

If we take the Laplace transform of $y(x,t)$ and integrate over R , we obtain from Eq. (4.16) the Laplace transform of $\langle \psi_1(t) \rangle$. However, we see from Eq. (4.18) that this is just $G(s)$. Hence

$$L \{ \langle \psi_1(t) \rangle \} = G(s). \quad (4.22)$$

The above can be evaluated by using the expression seen in Eq. (4.21). We will now analyze in depth the special case when

$$c_0 \pi(x) = a U(-b,b) \quad (4.23)$$

where $U(-b, b)$ is a rectangular function of unit height extending from $-b$ to b , and a is a constant proportional to the jump rate. From Eq. (4.23) we see that the transition p.d.f. is uniform so that

$$\pi(x) = U(-b, b) \frac{1}{2b} \quad (4.24)$$

and

$$c_0 = 2ab. \quad (4.25)$$

Dividing $U(-b, b)$ by $2b$ normalizes the density function. The region R now extends from $-b$ to b . Essentially Eq. (4.24) implies that the next state of the stochastic process, $X(t)$, is chosen from a uniform p.d.f. on the interval $(-b, b)$, and that this selected state is independent of all previous values of $X(t)$. If we set

$$\Omega(s) = \int_{-b}^b \frac{dx}{s + \frac{1}{\tau} + 2ab + g \frac{N}{\tau} x^2}, \quad (4.26)$$

then by algebraic manipulation, we can show that Eq. (4.22) becomes

$$L\{\langle \psi_1(t) \rangle\} = \frac{1}{2bN} \frac{\Omega(s)}{1 - a\Omega(s)} - \frac{1}{N} \frac{1}{s + \frac{1}{\tau}} \quad (4.27)$$

or equivalently

$$\langle \psi_1(t) \rangle = \frac{1}{2bN} L^{-1} \left\{ \frac{\Omega(s)}{1 - a\Omega(s)} \right\} - \frac{1}{N} e^{-\frac{t}{\tau}}. \quad (4.28)$$

C. Time Constants

In this section, we examine the Laplace transform solution of $\langle \psi_1(t) \rangle$ in detail. Recall from Section II, that measures of effectiveness of the adaptive filter are the time constants or settling times of the adapting weights $W(t)$. These time constants are related to the Laplace transform of $\langle W(t) \rangle$, which in turn is related to the Laplace transform of $\langle \psi_1(t) \rangle$ through Eqs. (4.10) and (4.4). From Eq. (4.27) it is apparent that the poles of $L\{\langle \psi_1(t) \rangle\}$ other than $s = -1/\tau$ will be dependent on the function $\Omega(s)$. A discussion of the function $\Omega(s)$ follows. It can be shown that $\Omega(s)$ is a single valued function of the complex variable s and has a line of singularities extending along an interval on the negative real axis as seen in Fig. 5.

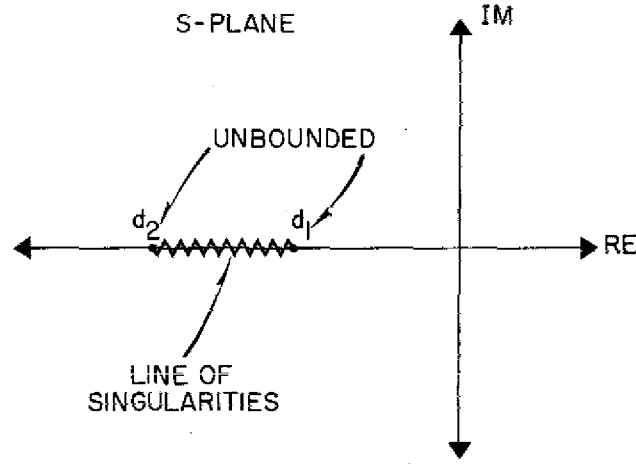
If we set

$$d_1 = -\frac{1}{\tau} - 2ab; \quad d_2 = -\frac{1}{\tau} - 2ab - \frac{b^2}{r^2} \quad (4.29)$$

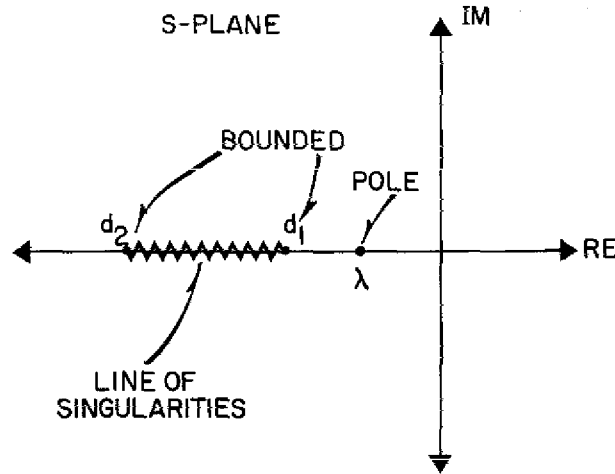
where $r = (\tau/gN)^{1/2}$, then this interval is $[d_2, d_1]$. The function $\Omega(s)$ is unbounded at the end points of this interval and we can also show that if $x \in (d_2, d_1)$, then

$$\lim_{\epsilon \rightarrow 0^+} \Omega(x + j\epsilon) = \frac{r}{y^{1/2}} \ln \left[\frac{b - ry^{1/2}}{b + ry^{1/2}} \right] - j \frac{\pi r}{y^{1/2}} \quad (4.30)$$

$$\lim_{\epsilon \rightarrow 0^+} \Omega(x - j\epsilon) = \frac{r}{y^{1/2}} \ln \left[\frac{b - ry^{1/2}}{b + ry^{1/2}} \right] + j \frac{\pi r}{y^{1/2}} \quad (4.31)$$

Fig. 5 — The function $\Omega(s)$

where $y = d_1 - x$. Equations (4.30) and (4.31) demonstrate the discontinuous nature of $\Omega(s)$ on the interval $[d_2, d_1]$. The interested reader is referred to Ref. 22 for proofs of these properties. The function $\Omega(s) [1 - a\Omega(s)]^{-1}$, which is part of the solution of $L\{\langle\psi_1(t)\rangle\}$, will also have a line of singularities extending between the same limits as seen in Fig. 6.

Fig. 6 — The function $\Omega(s) [1 - a\Omega(s)]^{-1}$

It is seen that the time constants of $\psi_1(t)$ and the weighting vector of the Applebaum algorithm will be dependent on the poles of the function $\Omega(s) [1 - a\Omega(s)]^{-1}$. It can be shown that this function has only one pole, λ , which satisfies the equation

$$a\Omega(\lambda) = 1. \quad (4.32)$$

It can be shown that this pole is real and is always located on the interval

$$\left[d_1, -\frac{1}{\tau} \right].$$

Thus the pole is never on the line of singularities as seen in Fig. 6. To show that $\lambda < -1/\tau$, we find by using Eq. (4.32) and the definition of $\Omega(s)$, Eq. (4.26), that

$$a\Omega(\lambda) - 1 = \int_{-b}^b \frac{\frac{gN}{\tau} x^2 + \frac{1}{\tau} + \lambda}{\frac{gN}{\tau} x^2 - d_1 + \lambda} dx = 0. \quad (4.33)$$

Assume $\lambda \geq -1/\tau$, then the numerator and denominator of the integral's kernel seen in Eq. (4.33) are always greater than zero. Hence, the integral is always greater than zero and Eq. (4.33) is never true. Thus, $\lambda < -1/\tau$. We can use Eqs. (4.30) and (4.31) to show that λ does not lie on the interval $[d_2, d_1]$. From Eqs. (4.30) and (4.31), the function $\Omega(s)$ has values that have imaginary parts greater than zero as we come arbitrarily close to the interval $[d_2, d_1]$. Hence, along this interval, $a\Omega(\lambda) - 1$ will have a nonzero imaginary part and cannot equal zero. We can also use Eq. (4.33) to show that λ is not less than d_2 . If this were true, then the numerator and denominator of integral's kernel in Eq. (4.33) would both be negative. Hence, the kernel is always positive which implies that $a\Omega(\lambda) - 1 > 0$. Thus, if λ cannot be less than d_2 and $\lambda \notin [d_2, d_1]$, then $\lambda > d_1$.

It should be noted that although $\Omega(s)$ is unbounded at d_1 and d_2 , that $\Omega(s)/(1 - a\Omega(s))$ is bounded. It can be shown by using elementary limiting procedures that

$$\lim_{s \rightarrow d_i} \frac{\Omega(s)}{1 - a\Omega(s)} = \frac{1}{a}; \quad i = 1, 2. \quad (4.34)$$

Integrating Eq. (4.26), we obtain

$$\Omega(\lambda) = \frac{2r}{\left[\lambda + \frac{1}{\tau} + 2ab\right]^{1/2}} \arctan \left\{ \frac{b}{r\left[\lambda + \frac{1}{\tau} + 2ab\right]^{1/2}} \right\} \quad (4.35)$$

where $\arctan \{\cdot\}$ is the principle branch of the inverse tangent function.

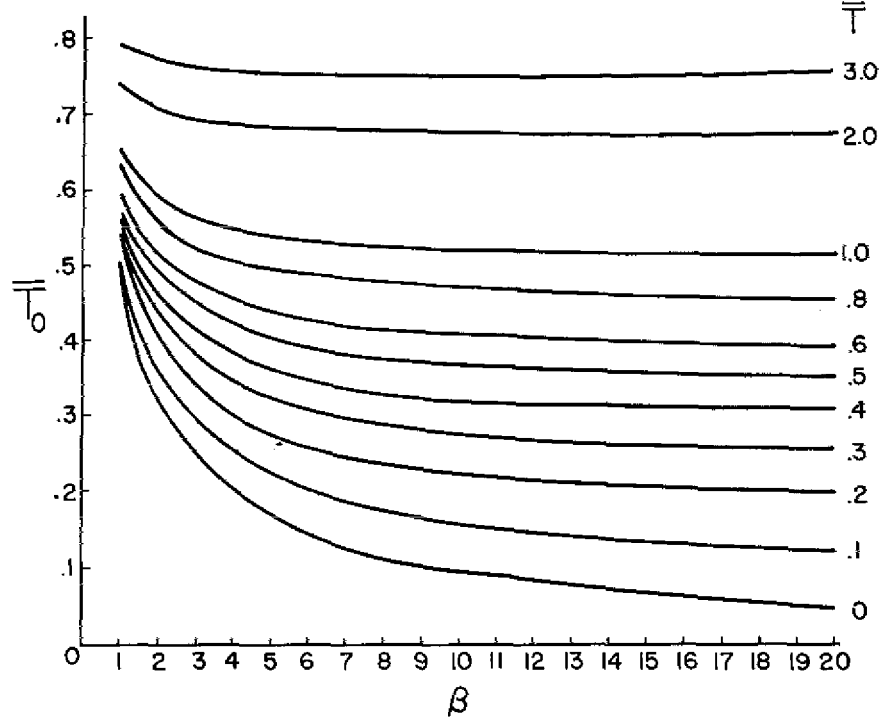
We will define $T_0 = -1/\lambda$ as the output time constant of the adaptive process and $\beta = gN\langle X^2 \rangle$. The parameter β will be called the element-gain-input noise power product because it is the multiplicative product of the number of elements in the receiving antenna's array, the gain of the receiver's amplifiers, and the input noise power measured at the receiver's front end.

To better understand the nature of the adaptive processor's time constant let us substitute the expression for $\Omega(\lambda)$ given in Eq. (4.35) into Eq. (4.32) and rewrite Eq. (4.32) as a function of β and the following normalized parameters: $\bar{T} = T/\tau$, and $\bar{T}_0 = T_0/\tau$ where T is the average switching time of the input noise ($T = 1/c_0$). This equation becomes

$$\frac{1}{[3\beta \bar{T}^2(1 - \bar{T}_0^{-1} + \bar{T}^{-1})]^{1/2}} \arctan \left[\frac{3\beta}{1 - \bar{T}_0^{-1} + \bar{T}^{-1}} \right]^{1/2} = 1. \quad (4.36)$$

The parameters \bar{T} and \bar{T}_0 will be called the normalized average switching time and the normalized time constant of the adaptive output process, respectively. Both are normalized to the time constant associated with the adaptive receiver's integrators seen in Fig. 2. If we plot \bar{T}_0 versus β for various contours of \bar{T} by using Eq. (4.36), we obtain curves as seen in Fig. 7. Note that \bar{T}_0 appears to be a monotonically decreasing function of β and a monotonically increasing function of \bar{T} . Using the property that $d_2 < \lambda < -1/\tau$, we can show that

$$\frac{\bar{T}}{1 + \bar{T}} < \bar{T}_0 < 1, \quad (4.37)$$

Fig. 7 — Normalized time constant, \bar{T}_0 , vs \bar{T} and β

and that as $\beta \rightarrow \infty$, the normalized time constant, \bar{T}_0 , approaches the lower bound of the above inequality. It is significant to note that past researchers [11] have derived \bar{T}_0 versus β for when $\bar{T} = 0$. We see from Fig. 7 that this assumption that the input noise is varying infinitely fast results in an upper bound on the performance in terms of minimizing the time constant of the adaptive processor. Also, past researchers [11,14] implied that as the input noise power becomes large (or $\beta \rightarrow \infty$) that the output time constant goes to zero. We see from Fig. 7 that this is not the case for nonzero \bar{T} . As the input noise power becomes large, the normalized time constant approaches a lower bound equal to $\bar{T}/(1 + \bar{T})$.

To give the reader some practical insight into the meaning of the time constant curves seen in Fig. 7, consider the example illustrated in Fig. 8. Here, we show (Fig. 8(a)) a sample realization of the amplitude modulation noise, $X(t)$, which is a CSJMP with $\bar{T} = 1$. Figure 8(b) is sample plot of the magnitude of the i th element of the output weighting vector, $W_i(t)$. Initially, $W_i(t)$ is in the quiescent state. When a noise step is applied at $t = t_0$, $W_i(t)$ adjusts itself to null out the noise source. At $t = t_1$, the input noise power is decreased and again the weight element adjusts itself. Note that the time constant $T_0(t_0)$ is greater than $T_0(t_1)$. This can be explained by using Fig. 7. Because the input noise power is greater at t_0 than at t_1 , we see from Fig. 7 that $\bar{T}_0(t_0) > \bar{T}_0(t_1)$ and hence $T_0(t_0) > T_0(t_1)$. The weight $W_i(t)$ will approach its optimum value after some period of time. The curves of Fig. 7 imply that it will reach this optimum value (with arbitrary accuracy) faster if the input noise power is larger. In addition, if the fluctuations are more rapid (say $\bar{T} = 0.5$), $W_i(t)$ will reach its optimum value faster.

If we expand \bar{T}_0 in a Taylor series as a function of small \bar{T} , we can show that [22]

$$\bar{T}_0 = \frac{1}{1 + \beta} \left[1 + \frac{4}{5} \frac{\beta^2}{1 + \beta} \bar{T} + O(\bar{T}^2) \right]. \quad (4.38)$$

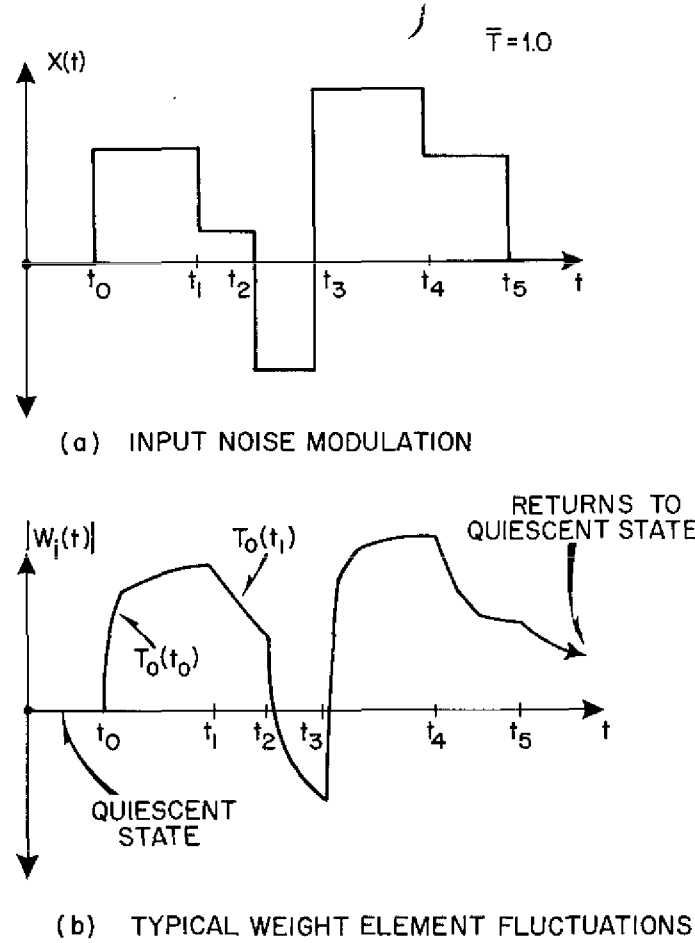


Fig. 8 — Input/output example

It is seen from Eq. (4.38) that \bar{T}_0 can be approximated to a first order term in \bar{T} if

$$T \ll \frac{\beta + 1}{\beta^2} \tau. \quad (4.39)$$

Equation (4.39) also specifies the bounds of the validity of the often used assumption that the input noise process is varying much faster than the output process. For $\beta \gg 1$, this assumption is valid when $T \ll \tau/\beta$ or equivalently when the input noise bandwidth $B_{in} \gg gN \langle X \rangle^2 / (\pi \tau)$.

From Eq. (4.38), we see that \bar{T}_0 and hence the time constant of the output process increases for small T . In fact, it is shown in Ref. 22 that the time constant is a minimum as $T \rightarrow 0$ or equivalently when the input process varies infinitely fast.

If the input noise is varying infinitely fast or $\bar{T} = 0$, we see from Eq. (4.38) that $\bar{T}_0 = 1/(1 + \beta)$. It is interesting to compare this result if the normalized time constant is acquired by the methodology used in Section II. If we take the expected value of both sides of Eq. (4.12) and assume that the input is varying much faster than the output such that $\langle X^2 \psi_1 \rangle = \langle X^2 \rangle \langle \psi_1 \rangle$, then

$$\frac{d\langle \psi_1 \rangle}{dt} = - \left[\frac{gN}{\tau} \langle X^2 \rangle + \frac{1}{\tau} \right] \langle \psi_1(t) \rangle + \frac{g}{\tau} \langle X^2 \rangle e^{-t/\tau}. \quad (4.40)$$

From Eq. (4.40), it is apparent that the eigenvalue is

$$\lambda = -\frac{gN}{\tau} < X^2 > - \frac{1}{\tau}, \quad (4.41)$$

and hence the normalized time constant is $1/(1 + \beta)$. This agrees with the result obtained by our previous methodology.

D. Exact Solution for First Moment

In this subsection, we obtain an exact expression for the first moment $\langle W(t) \rangle$ of the weighting vector for our specific example: adaptive processing when there is only one external noise source which generates a pseudo-white CSJMP. We examine $\langle W(t) \rangle$ in the steady state ($t \rightarrow \infty$) and compare this result with the solution that is obtained using the assumption that the input noise is fluctuating much faster than the output process, $W(t)$.

We now find the first moment $\langle W(t) \rangle$ of the weighting vector by using Eq. (4.28). To start with, the complex function $\Omega(s) [1 - a\Omega(s)]^{-1}$ has a line of singularities on the negative real axis; thus part of the solution of $\langle \psi_1(t) \rangle$ will be in the form of a real integral. Reference 22 evaluates the inverse Laplace transform of $\Omega(s) [1 - a\Omega(s)]^{-1}$ in terms of τ , a , N , g , and b . If we substitute for these parameters the defined parameters \bar{T} , β , and \bar{T}_0 , then $L^{-1} \{ \Omega(s) [1 - a\Omega(s)]^{-1} \}$ may be shown to be

$$L^{-1} \{ \Omega(s) [1 - a\Omega(s)]^{-1} \} = 2b \left[e^{-\frac{t}{\bar{T}_0\tau}} + (6\beta\bar{T})^2 e^{-(1+\bar{T}^{-1})\frac{t}{\tau}} \int_0^1 \frac{\omega^2 e^{-3\beta\omega^2\frac{t}{\tau}} d\omega}{\left[6\beta\bar{T} - \ln \left(\frac{1-\omega}{1+\omega} \right) \right]^2 + \pi^2} \right]. \quad (4.42)$$

The second term in the large brackets is due to line of singularities on the $[d_2, d_1]$. Note that as $\bar{T} \rightarrow 0$, the second term goes to 0. Thus, the integral seen in Eq. (4.42) does not contribute to $\langle \psi_1(t) \rangle$ when the input is fluctuating infinitely fast.

Our original intentions were to find the first moment $\langle W(t) \rangle$ of the adapting weight vector. This can now be done by substituting the expression for $L^{-1} \{ \Omega(s) [1 - a\Omega(s)]^{-1} \}$ seen in Eq. (4.42) into Eq. (4.28) which yields a complete solution for $\langle \psi_1(t) \rangle$. Then the first moment $\langle \Psi(t) \rangle$ of the transition matrix can be found by using Eq. (4.10) and recognizing that $\langle \psi_2(t) \rangle = e^{-t/\tau}$. This expression for $\langle \Psi(t) \rangle$ is then substituted into Eq. (4.4) which yields

$$\begin{aligned} \langle W(t) \rangle = & \left\{ \frac{1}{N} A^* A_T \left[e^{-\frac{t}{\tau}} - 1 + \bar{T}_0 (1 - e^{-\frac{t}{\bar{T}_0\tau}}) + (6\beta\bar{T})^2 \int_0^1 \frac{\omega^2 (1 - e^{-r_1(\omega)t})}{r_1(\omega)r_2(\omega)} d\omega \right] \right. \\ & \left. + I(1 - e^{-\frac{t}{\tau}}) \right\} S^* g \end{aligned} \quad (4.43)$$

where

$$r_1(\omega) = 3\beta\omega^2 + 1 + \bar{T}^{-1} \quad (4.44)$$

$$r_2(\omega) = \left[6\beta\bar{T} - \ln \left(\frac{1-\omega}{1+\omega} \right) \right]^2 + \pi^2. \quad (4.45)$$

If we consider the steady state value of the weighting vector and denote this by $\bar{W}_\infty = \lim_{t \rightarrow \infty} \langle W(t) \rangle$, then from Eq. (4.43) it is seen that

$$\bar{W}_\infty = \left[\frac{1}{N} A^* A_T \left(\bar{T}_0 - 1 + (6\beta \bar{T})^2 \int_0^1 \frac{\omega^2 d\omega}{r_1(\omega) r_2(\omega)} \right) + I \right] S^* g \quad (4.46)$$

where $\bar{T}_0 = \bar{T}_0(\beta, \bar{T})$. Had we derived \bar{W}_∞ under the assumption that the input $X(t)$ was fluctuating much faster than the output $W(t)$ then it would be found as shown in Ref. 22 that

$$\bar{W}_\infty = \left[\frac{1}{N} A^* A_T (\bar{T}_0 - 1) + I \right] S^* g \quad (4.47)$$

where $\bar{T}_0 = \bar{T}_0(\beta, 0) = 1/(\beta + 1)$. Equation (4.47) was found by taking the expected value of both sides of Eq. (4.1), setting the time derivative equal to zero, and solving for \bar{W}_∞ by using the matrix inversion lemma [26] and the assumption that $\langle X^2(t) W(t) \rangle = \langle X^2(t) \rangle \langle W(t) \rangle$. We observe that Eq. (4.46) differs from the classical result, Eq. (4.47), by (1) the contribution of the line of singularities and (2) by the fact that $\bar{T}_0(\beta, \bar{T}) > \bar{T}_0(\beta, 0)$ for $\bar{T} \neq 0$. However, for $\bar{T} = 0$, Eq. (4.46) is identical to Eq. (4.47).

V. SUMMARY AND CONCLUSIONS

In this report, we have introduced a technique that yields closed formed solutions for the noise analysis of the Applebaum adaptive algorithm. Past researchers have derived results under the assumption that the input noise is stationary and rapidly varying with respect to the output weighting vector of the Applebaum algorithm. We have extended this work to include noise that is not stationary and can vary at any rate. The technique models the external noise sources as continuous state jump Markov processes that modulate a carrier frequency. The first moment equation for the optimal weighting vector that results is an integro-differential equation that is solvable in specific cases. In particular, we have examined in detail the case where the adaptive algorithm is subjected to a single external noise source whose modulation is a continuous state jump Markov process. We found that the assumption made by past researchers [3,13,14] that the input noise is rapidly varying with respect to weighting vector imposes an upper bound on performance. It is shown in this case that the time constant of the adaptive output process will be minimized. As the input noise process variations become slower, the time constant of the output process becomes larger. Bounds on the validity of the above assumption are established in Section IV. Also past researchers [3,13,14] implied that if the input noise power became infinite, then the output time constant would go to zero. We show that for input noise with a finite bandwidth, that the output time constant approaches a positive lower bound as the input noise power becomes finite.

VI. REFERENCES

1. R.E. Kalman and N. DeClaris, eds., *Aspects of Network and System Theory*, Holt, Rinehart, and Winston Inc., 1971.
2. P.W. Howells, "Intermediate Frequency Side-Lobe Canceller," U.S. Patent 3,303,990, August 24, 1965 (filed May 4, 1959).
3. S.P. Applebaum, "Adaptive Arrays," *IEEE Trans. on Antennas and Propagation*, **24**(5), 585-598, September 1976; also Syracuse Univ. Res. Corp., Rep. SPL TR 66-1, August 1966.

4. B. Widrow, "Adaptive Filters I: Fundamentals," Stanford Univ. Electronics Labs., Syst. Theory Lab., Center for Syst. Res. Rep. SU-SEL-66-126, Tech. Rep. 6764-6, December 1966.
5. B. Widrow, P.E. Mantey, L.J. Griffiths, and B.B. Goode, "Adaptive Antenna Systems," *Proc. IEEE*, **55**, 2143-2159, December 1967.
6. L.J. Griffiths, "A Simple Adaptive Algorithm for Real-Time Processing in Antenna Arrays," *Proc. IEEE*, **57**, 1696-1704, October 1969.
7. O.L. Frost III, "An Algorithm for Linearly Constrained Adaptive Array Processing," *Proc. IEEE*, **60**, 926-935, August 1972.
8. R.L. Riegler and R.T. Compton, Jr., "An Adaptive Array for Interference Rejection," *Proc. IEEE*, **61**, 748-758, June 1973.
9. R.T. Compton, Jr., "Adaptive Arrays: On Power Equalization with Proportional Control," Ohio State Univ. Electroscience Lab., Rep. 3234-1, Contract N00019-71-C-0219, December 1971.
10. C.L. Zahm, "Applications of Adaptive Arrays to Suppress Strong Jammers in the Presence of Weak Signals," *IEEE Trans. Aerosp. Electron. Syst.*, **AES-9**, 260-271, March 1973.
11. L.E. Brennan, E.L. Pugh, and I.S. Reed, "Control-Loop Noise in Adaptive Array Antennas," *IEEE Trans. Aerosp. Electron. Syst.*, **AES-7**, 254-262, March 1971.
12. L.E. Brennan and I.S. Reed, "Effect of Envelope Limiting in Adaptive Array Control Loops," *IEEE Trans. Aerosp. Electron. Syst.*, **AES-7**, 698-700, July 1971.
13. L.E. Brennan and I.S. Reed, "Theory of Adaptive Radar," *IEEE Trans. Aerosp. Electron. Syst.*, **AES-9**, 237-252, March 1973.
14. W.F. Gabriel, "An Introduction to Adaptive Arrays," *Proc. IEEE*, **64**, 239-272, February 1976.
15. N.J. Bershad, "Performance Analysis of Adaptive Processors via Diffusion Model Approximations and the Fokker-Planck Equation," University of Calif. at Irvine, Stochastic Systems Research Report ONR-76-1, March 1976.
16. W. Feller, "On the Integro-Differential Equations of Purely Discontinuous Markov Processes," *Trans. Amer. Math. Soc.* **48**, 488-515, 1940.
17. U. Frisch, "Wave Propagation in Random Media," in *Probabilistic Methods in Appl. Math.*, Academic Press, New York, 1968, Vol. 1, pp. 96-100.
18. A. Kolmogorov, "Über die Analytischen Methoden in der Wahrscheinlichkeitsrechnung," *Math. Ann.* **104**, 415-458, 1931.
19. S.K. Srinivasan, *Stochastic Point Processes*, Hafner Press, 1974, pp. 79-83.
20. A.T. Bharucha-Reid, *Elements of the Theory of Markoff Processes and Their Applications*, McGraw Hill Inc., 1960, pp. 57-128.
21. N.U. Prabhu, *Stochastic Processes*, The MacMillan Co., 1965, pp. 112-124.

22. K.R. Gerlach, "Application of Probabilistic Methods in Adaptive Arrays," D.Sc. dissertation, George Washington University, February 1981.
23. C.T. Chen, *Introduction to Linear System Theory*, Appendix D, Holt, Rinehart, and Winston Inc., 1970.
24. R.H. Lang and R. Pickholtz, "Novel ECCM Techniques for Army Tactical Communications," Report TCA-79-2-003, CORACOM, Fort Monmouth, N.J., June 1979. (Lang and Pickholtz contributed Chapter 4).
25. R.M. Wilcox, "Exponential Operators and Parameter Differentiation in Quantum Physics," *J. Math. Phys.*, 8(4), 962-982.
26. A.P. Sage and J.L. Melsa, *Estimation Theory with Applications to Communications and Control*, Appendix A, McGraw Hill, 1971.